Formulae involving ∇
Vector Identities with Proofs: Nabla Formulae for Vector Analysis

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Vector: $\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$ $\quad\mathbf{B} = B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k}$ $\quad\mathbf{C} = C_1\,\mathbf{i} + C_2\,\mathbf{j} + C_3\,\mathbf{k}$

Scalar: $\phi = \phi(x,y,z)$ $\quad\psi = \psi(x,y,z)$

Nabla: $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

(1) $(\mathbf{A} \times \mathbf{B}).\mathbf{C} \equiv (\mathbf{B} \times \mathbf{C}).\mathbf{A} \equiv (\mathbf{C} \times \mathbf{A}).\mathbf{B}$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \equiv (\mathbf{A}.\mathbf{C})\mathbf{B} - (\mathbf{A}.\mathbf{B})\mathbf{C}$

(3) Prove $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

$$\left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(\phi + \psi) = \frac{\partial(\phi+\psi)}{\partial x}i + \frac{\partial(\phi+\psi)}{\partial y}j + \frac{\partial(\phi+\psi)}{\partial z}k$$

$$= \frac{\partial\phi}{\partial x}i + \frac{\partial\psi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\psi}{\partial y}j + \frac{\partial\phi}{\partial z}k + \frac{\partial\psi}{\partial z}k$$

$$= \left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right) + \left(\frac{\partial\psi}{\partial x}i + \frac{\partial\psi}{\partial y}j + \frac{\partial\psi}{\partial z}k\right)$$

$$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\phi + \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\psi$$

$$\therefore \quad \nabla(\phi + \psi) = \nabla\phi + \nabla\psi$$

(4) Prove $\nabla(\phi\,\psi) = \phi\,\nabla\psi + \psi\,\nabla\phi$

$$\left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(\phi\psi) = \frac{\partial(\phi\psi)}{\partial x}i + \frac{\partial(\phi\psi)}{\partial y}j + \frac{\partial(\phi\psi)}{\partial z}k$$

$$= \phi\frac{\partial\psi}{\partial x}i + \psi\frac{\partial\phi}{\partial x}i + \phi\frac{\partial\psi}{\partial x}j + \psi\frac{\partial\phi}{\partial x}j + \phi\frac{\partial\psi}{\partial x}k + \psi\frac{\partial\phi}{\partial x}k$$

$$= \phi\left(\frac{\partial\psi}{\partial x}i + \frac{\partial\psi}{\partial y}j + \frac{\partial\psi}{\partial z}k\right) + \psi\left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right)$$

$$= \phi\left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\psi + \psi\left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\phi$$

$\therefore \quad \nabla(\phi\,\psi) = \phi\nabla\psi + \psi\nabla\phi$

(5) Prove $\nabla.(\mathbf{A} + \mathbf{B}) = \nabla.\mathbf{A} + \nabla.\mathbf{B}$

$$\nabla.(A+B) = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\left[(A_1+B_1)i + (A_2+B_2)j + (A_3+B_3)k\right]$$

$$= \frac{\partial(A_1+B_1)}{\partial x} + \frac{\partial(A_2+B_2)}{\partial y} + \frac{\partial(A_3+B_3)}{\partial z} = \text{LHS}$$

$$\nabla.A + \nabla.B = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(A_1 i + A_2 j + A_3 k) + \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(B_1 i + B_2 j + B_3 k)$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}$$

$$= \frac{\partial(A_1+B_1)}{\partial x} + \frac{\partial(A_2+B_2)}{\partial y} + \frac{\partial(A_3+B_3)}{\partial z} = \text{RHS}$$

LHS = RHS

$\therefore \quad \nabla.(\mathbf{A} + \mathbf{B}) = \nabla.\mathbf{A} + \nabla.\mathbf{B}$

(6) Prove $\nabla\mathrm{x}(\mathbf{A} + \mathbf{B}) = \nabla\mathrm{x}\mathbf{A} + \nabla\mathrm{x}\mathbf{B}$

$$\nabla x(A+B) = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)x\left[(A_1+B_1)i + (A_2+B_2)j + (A_3+B_3)k\right]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1+B_1 & A_2+B_2 & A_3+B_3 \end{vmatrix}$$

$$= \left(\frac{\partial(A_3+B_3)}{\partial y} - \frac{\partial(A_2+B_2)}{\partial z}\right)i - \left(\frac{\partial(A_3+B_3)}{\partial x} - \frac{\partial(A_1+B_1)}{\partial z}\right)j + \left(\frac{\partial(A_2+B_2)}{\partial x} - \frac{\partial(A_1+B_1)}{\partial y}\right)k$$

$$= \left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)i-\left(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}\right)j+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)k\right]+\left[\left(\frac{\partial B_3}{\partial y}-\frac{\partial B_2}{\partial z}\right)i-\left(\frac{\partial B_3}{\partial x}-\frac{\partial B_1}{\partial z}\right)j+\left(\frac{\partial B_2}{\partial x}-\frac{\partial B_1}{\partial y}\right)k\right]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$\therefore \quad \nabla \mathrm{x}(\mathbf{A}+\mathbf{B}) = \nabla \mathrm{x}\mathbf{A} + \nabla \mathrm{x}\mathbf{B}$

(7) Prove $\nabla.(\phi\mathbf{A}) = (\nabla\phi).\mathrm{A} + \phi(\nabla.\mathbf{A})$

$$\nabla.(\phi A) = \left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right).(\phi A_1 i+\phi A_2 j+\phi A_3 k)$$

$$= \frac{\partial(\phi A_1)}{\partial x}+\frac{\partial(\phi A_2)}{\partial y}+\frac{\partial(\phi A_3)}{\partial z} = \text{LHS}$$

$$(\nabla\phi).A+\phi(\nabla.A) = \left(\frac{\partial\phi}{\partial x}i+\frac{\partial\phi}{\partial y}j+\frac{\partial\phi}{\partial z}k\right).(A_1 i+A_2 j+A_3 k)+\phi\left[\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right).(A_1 i+A_2 j+A_3 k)\right]$$

$$= \left(A_1\frac{\partial\phi}{\partial x}+A_2\frac{\partial\phi}{\partial y}+A_3\frac{\partial\phi}{\partial z}\right)+\phi\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)$$

$$= \left(A_1\frac{\partial\phi}{\partial x}+\phi\frac{\partial A_1}{\partial x}\right)+\left(A_2\frac{\partial\phi}{\partial y}+\phi\frac{\partial A_2}{\partial y}\right)+\left(A_3\frac{\partial\phi}{\partial z}+\phi\frac{\partial A_3}{\partial z}\right)$$

$$= \frac{\partial(\phi A_1)}{\partial x}+\frac{\partial(\phi A_2)}{\partial y}+\frac{\partial(\phi A_3)}{\partial z} = \text{RHS}$$

LHS = RHS

$\therefore \quad \nabla.(\phi\mathbf{A}) = (\nabla\phi).\mathrm{A} + \phi(\nabla.\mathbf{A})$

(8) Prove $\nabla\mathrm{x}(\phi\mathbf{A}) = (\nabla\phi)\mathrm{x}\mathrm{A} + \phi(\nabla\mathrm{x}\mathbf{A})$

$$\nabla x(\phi A) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$= \left(\frac{\partial(\phi A_3)}{\partial y}-\frac{\partial(\phi A_2)}{\partial z}\right)i-\left(\frac{\partial(\phi A_3)}{\partial x}-\frac{\partial(\phi A_1)}{\partial z}\right)j+\left(\frac{\partial(\phi A_2)}{\partial x}-\frac{\partial(\phi A_1)}{\partial y}\right)k$$

$$= \left(\phi\frac{\partial A_3}{\partial y}+A_3\frac{\partial\phi}{\partial y}-\phi\frac{\partial A_2}{\partial y}-A_2\frac{\partial\phi}{\partial y}\right)i-\left(\phi\frac{\partial A_3}{\partial x}+A_3\frac{\partial\phi}{\partial x}-\phi\frac{\partial A_1}{\partial z}-A_1\frac{\partial\phi}{\partial z}\right)j+\left(\phi\frac{\partial A_2}{\partial x}+A_2\frac{\partial\phi}{\partial x}-\phi\frac{\partial A_1}{\partial y}-A_1\frac{\partial\phi}{\partial y}\right)k$$

$$= \left[\left(A_3\frac{\partial\phi}{\partial y}-A_2\frac{\partial\phi}{\partial y}\right)i-\left(A_3\frac{\partial\phi}{\partial x}-A_1\frac{\partial\phi}{\partial z}\right)j+\left(A_2\frac{\partial\phi}{\partial x}-A_1\frac{\partial\phi}{\partial y}\right)k\right]$$

$$+\left[\left(\phi\frac{\partial A_3}{\partial y}-\phi\frac{\partial A_2}{\partial y}\right)i-\left(\phi\frac{\partial A_3}{\partial x}-\phi\frac{\partial A_1}{\partial z}\right)j+\left(\phi\frac{\partial A_2}{\partial x}-\phi\frac{\partial A_1}{\partial y}\right)k\right]$$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}+\phi\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$\therefore \quad \nabla\text{x}(\phi\mathbf{A}) = (\nabla\phi)\text{xA} + \phi(\nabla\text{x}\mathbf{A})$

(9) Prove $\nabla.(\mathbf{A}\text{x}\mathbf{B}) = \mathbf{B}.(\nabla\text{x}\mathbf{A}) - \mathbf{A}.(\nabla\text{x}\mathbf{B})$

$$\nabla.(AxB)=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right).\begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right).[(A_2B_3-A_3B_2)i-(A_1B_3-A_3B_1)j+(A_1B_2-A_2B_1)k]$$

$$=\frac{\partial(A_2B_3-A_3B_2)}{\partial x}-\frac{\partial(A_1B_3-A_3B_1)}{\partial y}+\frac{\partial(A_1B_2-A_2B_1)}{\partial z}$$

$$B.(\nabla xA)=(B_1i+B_2j+B_3k).\left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)i-\left(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}\right)j+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)k\right]$$

$$=B_1\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)-B_2\left(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}\right)+B_3\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)$$

Similarly, by interchanging the variable of **A** and **B**, we have

$$A.(\nabla xB)=(A_1i+A_2j+A_3k).\left[\left(\frac{\partial B_3}{\partial y}-\frac{\partial B_2}{\partial z}\right)i-\left(\frac{\partial B_3}{\partial x}-\frac{\partial B_1}{\partial z}\right)j+\left(\frac{\partial B_2}{\partial x}-\frac{\partial B_1}{\partial y}\right)k\right]$$

$$=A_1\left(\frac{\partial B_3}{\partial y}-\frac{\partial B_2}{\partial z}\right)-A_2\left(\frac{\partial B_3}{\partial x}-\frac{\partial B_1}{\partial z}\right)+A_3\left(\frac{\partial B_2}{\partial x}-\frac{\partial B_1}{\partial y}\right)$$

$$\mathbf{B}.(\nabla\text{x}\mathbf{A})-\mathbf{A}.(\nabla\text{x}\mathbf{B})=\left(B_1\frac{\partial A_3}{\partial y}+A_3\frac{\partial B_1}{\partial y}\right)-\left(B_1\frac{\partial A_2}{\partial z}+A_2\frac{\partial B_1}{\partial z}\right)-\left(B_2\frac{\partial A_3}{\partial x}+A_3\frac{\partial B_2}{\partial x}\right)$$

$$+\left(B_2\frac{\partial A_1}{\partial z}+A_1\frac{\partial B_2}{\partial z}\right)+\left(B_3\frac{\partial A_2}{\partial x}+A_2\frac{\partial B_3}{\partial x}\right)-\left(B_3\frac{\partial A_1}{\partial y}+A_1\frac{\partial B_3}{\partial y}\right)$$

$$=\frac{\partial(A_3B_1)}{\partial y}-\frac{\partial(A_2B_1)}{\partial z}-\frac{\partial(A_3B_2)}{\partial x}+\frac{\partial(A_1B_2)}{\partial z}+\frac{\partial(A_2B_3)}{\partial x}-\frac{\partial(A_1B_3)}{\partial y}$$

$$=\frac{\partial(A_2B_3-A_3B_2)}{\partial x}-\frac{\partial(A_1B_3-A_3B_1)}{\partial y}+\frac{\partial(A_1B_2-A_2B_1)}{\partial z}$$

$\therefore \quad \nabla.(\mathbf{A}\text{x}\mathbf{B}) = \mathbf{B}.(\nabla\text{x}\mathbf{A}) - \mathbf{A}.(\nabla\text{x}\mathbf{B})$

(10) Prove ∇x(**A**x**B**) = (**B**.∇)**A** - **B**(∇.**A**) - (**A**.∇)**B** + **A**(∇.**B**)

$$\nabla x(AxB) = \nabla x\begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \nabla x\left[(A_2B_3 - A_3B_2)i - (A_1B_3 - A_3B_1)j + (A_1B_2 - A_2B_1)k\right]$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_2B_3 - A_3B_2 & A_3B_1 - A_1B_3 & A_1B_2 - A_2B_1 \end{vmatrix}$$

$$= \left(\frac{\partial(A_1B_2 - A_2B_1)}{\partial y} - \frac{\partial(A_3B_1 - A_1B_3)}{\partial z}\right)i - \left(\frac{\partial(A_1B_2 - A_2B_1)}{\partial x} - \frac{\partial(A_2B_3 - A_3B_2)}{\partial z}\right)j + \left(\frac{\partial(A_3B_1 - A_1B_3)}{\partial x} - \frac{\partial(A_2B_3 - A_3B_2)}{\partial y}\right)k$$

= LHS

(B.∇)**A** - **B**(∇.**A**) = $\left(B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y} + B_3\frac{\partial}{\partial z}\right)(A_1i + A_2j + A_3k) - \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)(B_1i + B_2j + B_3k)$

$$= \left(B_2\frac{\partial A_1}{\partial y} + B_3\frac{\partial A_1}{\partial z} - B_1\frac{\partial A_2}{\partial y} - B_1\frac{\partial A_3}{\partial z}\right)i + \left(B_1\frac{\partial A_2}{\partial x} + B_3\frac{\partial A_2}{\partial z} - B_2\frac{\partial A_1}{\partial x} - B_2\frac{\partial A_3}{\partial z}\right)j + \left(B_1\frac{\partial A_3}{\partial x} + B_2\frac{\partial A_3}{\partial y} - B_3\frac{\partial A_1}{\partial x} - B_3\frac{\partial A_2}{\partial y}\right)k$$

Similarly, by interchanging the variable of **A** and **B**, we have

(A.∇)**B** - **A**(∇.**B**) = $\left(A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}\right)(B_1i + B_2j + B_3k) - \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)(A_1i + A_2j + A_3k)$

$$= \left(A_2\frac{\partial B_1}{\partial y} + A_3\frac{\partial B_1}{\partial z} - A_1\frac{\partial B_2}{\partial y} - A_1\frac{\partial B_3}{\partial z}\right)i + \left(A_1\frac{\partial B_2}{\partial x} + A_3\frac{\partial B_2}{\partial z} - A_2\frac{\partial B_1}{\partial x} - A_2\frac{\partial B_3}{\partial z}\right)j + \left(A_1\frac{\partial B_3}{\partial x} + A_2\frac{\partial B_3}{\partial y} - A_3\frac{\partial B_1}{\partial x} - A_3\frac{\partial B_2}{\partial y}\right)k$$

(B.∇)**A** - **B**(∇.**A**) - (**A**.∇)**B** + **A**(∇.**B**)

$$= \left[\left(B_2\frac{\partial A_1}{\partial y} + A_1\frac{\partial B_2}{\partial y}\right) + \left(B_3\frac{\partial A_1}{\partial z} + A_1\frac{\partial B_3}{\partial z}\right) - \left(B_1\frac{\partial A_2}{\partial y} + A_2\frac{\partial B_1}{\partial y}\right) - \left(B_1\frac{\partial A_3}{\partial z} + A_3\frac{\partial B_1}{\partial z}\right)\right]i$$

$$+ \left[\left(B_1\frac{\partial A_2}{\partial x} + A_1\frac{\partial B_2}{\partial x}\right) + \left(B_3\frac{\partial A_2}{\partial z} + A_2\frac{\partial B_3}{\partial z}\right) - \left(B_2\frac{\partial A_1}{\partial x} + A_1\frac{\partial B_2}{\partial x}\right) - \left(B_2\frac{\partial A_3}{\partial z} + A_3\frac{\partial B_2}{\partial z}\right)\right]j$$

$$+ \left[\left(B_1\frac{\partial A_3}{\partial x} + A_3\frac{\partial B_1}{\partial x}\right) + \left(B_2\frac{\partial A_3}{\partial y} + A_3\frac{\partial B_2}{\partial y}\right) - \left(B_3\frac{\partial A_1}{\partial x} + A_1\frac{\partial B_3}{\partial x}\right) - \left(B_3\frac{\partial A_2}{\partial y} + A_2\frac{\partial B_3}{\partial y}\right)\right]k$$

$$= \left(\frac{\partial(A_1B_2 - A_2B_1)}{\partial y} + \frac{\partial(A_1B_3 - A_3B_1)}{\partial z}\right)i - \left(\frac{\partial(A_1B_2 - A_2B_1)}{\partial x} + \frac{\partial(A_3B_2 - A_2B_3)}{\partial z}\right)j + \left(\frac{\partial(A_3B_1 - A_1B_3)}{\partial x} + \frac{\partial(A_3B_2 - A_2B_3)}{\partial y}\right)k$$

$$= \left(\frac{\partial(A_1B_2 - A_2B_1)}{\partial y} - \frac{\partial(A_3B_1 - A_1B_3)}{\partial z}\right)i - \left(\frac{\partial(A_1B_2 - A_2B_1)}{\partial x} - \frac{\partial(A_2B_3 - A_3B_2)}{\partial z}\right)j + \left(\frac{\partial(A_3B_1 - A_1B_3)}{\partial x} - \frac{\partial(A_2B_3 - A_3B_2)}{\partial y}\right)k$$

= RHS

RHS = LHS

$\therefore \quad \nabla x(\mathbf{A}x\mathbf{B}) = (\mathbf{B}.\nabla)\mathbf{A} - \mathbf{B}(\nabla.\mathbf{A}) - (\mathbf{A}.\nabla)\mathbf{B} + \mathbf{A}(\nabla.\mathbf{B})$

(11) Prove $\nabla(\mathbf{A.B}) = (\mathbf{B}.\nabla)\mathbf{A} + (\mathbf{A}.\nabla)\mathbf{B} + \mathbf{B}x(\nabla x\mathbf{A}) + \mathbf{A}x(\nabla x\mathbf{B})$

$$\nabla(\mathbf{A.B}) = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(A_1B_1 + A_2B_2 + A_3B_3) = \text{LHS}$$

$$(\mathbf{B}.\nabla)\mathbf{A} = \left(B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y} + B_3\frac{\partial}{\partial z}\right)(A_1i + A_2j + A_3k)$$

$$Bx(\nabla xA) = (B_1i + B_2j + B_3k)x\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)i - \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)j + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)k\right]$$

$$= \begin{vmatrix} i & j & k \\ B_1 & B_2 & B_3 \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \left[\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)B_2 - \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)B_3\right]i - \left[\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)B_1 - \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)B_3\right]j + \left[\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)B_1 - \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)B_2\right]k$$

Similarly, by interchanging the variable of **A** and **B**, we have

$$(\mathbf{A}.\nabla)\mathbf{B} = \left(A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}\right)(B_1i + B_2j + B_3k)$$

$$Ax(\nabla xB) = (A_1i + A_2j + A_3k)x\left[\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)i - \left(\frac{\partial B_3}{\partial x} - \frac{\partial B_1}{\partial z}\right)j + \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)k\right]$$

$$= \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ \frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} & \frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x} & \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} \end{vmatrix}$$

$$= \left[\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)A_2 - \left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right)A_3\right]i - \left[\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)A_1 - \left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)A_3\right]j + \left[\left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right)A_1 - \left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)A_2\right]k$$

Hence

$(\mathbf{B}.\nabla)\mathbf{A} + \mathbf{B}x(\nabla x\mathbf{A})$

$$= \left(B_1\frac{\partial A_1}{\partial x} + B_2\frac{\partial A_2}{\partial x} + B_3\frac{\partial A_3}{\partial x}\right)i + \left(B_2\frac{\partial A_2}{\partial y} + B_1\frac{\partial A_1}{\partial y} + B_3\frac{\partial A_3}{\partial y}\right)j + \left(B_3\frac{\partial A_3}{\partial z} + B_1\frac{\partial A_1}{\partial z} + B_2\frac{\partial A_2}{\partial z}\right)k$$

$(\mathbf{A}.\nabla)\mathbf{B} + \mathbf{A}x(\nabla x\mathbf{B})$

$$= \left(A_1\frac{\partial B_1}{\partial x} + A_2\frac{\partial B_2}{\partial x} + A_3\frac{\partial B_3}{\partial x}\right)i + \left(A_2\frac{\partial B_2}{\partial y} + A_1\frac{\partial B_1}{\partial y} + A_3\frac{\partial B_3}{\partial y}\right)j + \left(A_3\frac{\partial B_3}{\partial z} + A_1\frac{\partial B_1}{\partial z} + A_2\frac{\partial B_2}{\partial z}\right)k$$

$(\mathbf{B}.\nabla)\mathbf{A} + (\mathbf{A}.\nabla)\mathbf{B} + \mathbf{B}\text{x}(\nabla\text{x}\mathbf{A}) + \mathbf{A}\text{x}(\nabla\text{x}\mathbf{B})$

$$= \left(\frac{\partial(A_1B_1)}{\partial x} + \frac{\partial(A_2B_2)}{\partial x} + \frac{\partial(A_3B_3)}{\partial x}\right)i + \left(\frac{\partial(A_2B_2)}{\partial y} + \frac{\partial(A_1B_1)}{\partial y} + \frac{\partial(A_3B_3)}{\partial y}\right)j + \left(\frac{\partial(A_3B_3)}{\partial z} + \frac{\partial(A_1B_1)}{\partial z} + \frac{\partial(A_2B_2)}{\partial z}\right)k$$

$$= \frac{\partial(A_1B_1 + A_2B_2 + A_3B_3)}{\partial x}i + \frac{\partial(A_1B_1 + A_2B_2 + A_3B_3)}{\partial y}j + \frac{\partial(A_1B_1 + A_2B_2 + A_3B_3)}{\partial z}k$$

$$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(A_1B_1 + A_2B_2 + A_3B_3) = \text{RHS}$$

LHS = RHS

$\therefore \quad \nabla(\mathbf{A.B}) = (\mathbf{B}.\nabla)\mathbf{A} + (\mathbf{A}.\nabla)\mathbf{B} + \mathbf{B}\text{x}(\nabla\text{x}\mathbf{A}) + \mathbf{A}\text{x}(\nabla\text{x}\mathbf{B})$

(12) Prove $\nabla.(\nabla\phi) = \nabla^2\phi$

$$\nabla.(\nabla\phi) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi$$

$\therefore \quad \nabla.(\nabla\phi) = \nabla^2\phi$

(13) Prove $\nabla\text{x}(\nabla\phi) = 0$

$$\nabla\text{x}(\nabla\phi) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)x\left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= (\phi_{zy} - \phi_{yz})i - (\phi_{zx} - \phi_{xz})j + (\phi_{yx} - \phi_{xy})k$$

Since ϕ has continuous second order partial derivatives, we have

$\phi_{xy} = \phi_{yx} \qquad \phi_{yz} = \phi_{zy} \qquad \phi_{zx} = \phi_{xz}$

$\therefore \quad \nabla\text{x}(\nabla\phi) = 0$

(14) Prove $\nabla.(\nabla x\mathbf{A}) = 0$

$$\nabla.(\nabla x\mathbf{A}) = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)i-\left(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}\right)j+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)k\right]$$

$$= \left(\frac{\partial^2 A_3}{\partial y\partial x}-\frac{\partial^2 A_2}{\partial z\partial x}\right)-\left(\frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_1}{\partial z\partial y}\right)+\left(\frac{\partial^2 A_2}{\partial x\partial z}-\frac{\partial^2 A_1}{\partial y\partial z}\right)$$

$$= 0$$

$\therefore$ $\nabla.(\nabla x\mathbf{A}) = 0$

(15) Prove $\nabla x(\nabla x\mathbf{A}) = \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A}$

$$\nabla x(\nabla x\mathbf{A}) = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)x\left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)i-\left(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial z}\right)j+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)k\right]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \left(\frac{\partial^2 A_2}{\partial x\partial y}-\frac{\partial^2 A_1}{\partial y^2}-\frac{\partial^2 A_1}{\partial z^2}+\frac{\partial^2 A_3}{\partial z\partial x}\right)i-\left(\frac{\partial^2 A_2}{\partial x^2}-\frac{\partial^2 A_1}{\partial y\partial x}-\frac{\partial^2 A_3}{\partial y\partial z}+\frac{\partial^2 A_2}{\partial z^2}\right)j+\left(\frac{\partial^2 A_1}{\partial z\partial x}-\frac{\partial^2 A_3}{\partial x^2}-\frac{\partial^2 A_3}{\partial y^2}+\frac{\partial^2 A_2}{\partial z\partial y}\right)k$$

= LHS

$\nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A}$

$$= \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(A_1 i+A_2 j+A_3 k)$$

$$= \left(\frac{\partial^2 A_2}{\partial y\partial x}+\frac{\partial^2 A_3}{\partial z\partial x}-\frac{\partial^2 A_1}{\partial y^2}-\frac{\partial^2 A_1}{\partial z^2}\right)i+\left(\frac{\partial^2 A_1}{\partial x\partial y}+\frac{\partial^2 A_3}{\partial z\partial y}-\frac{\partial^2 A_2}{\partial x^2}-\frac{\partial^2 A_2}{\partial z^2}\right)j+\left(\frac{\partial^2 A_1}{\partial x\partial z}+\frac{\partial^2 A_2}{\partial y\partial z}-\frac{\partial^2 A_3}{\partial x^2}-\frac{\partial^2 A_3}{\partial y^2}\right)k$$

= RHS

LHS = RHS

$\therefore$ $\nabla x(\nabla x\mathbf{A}) = \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A}$